Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 1, 93–111

On the generic nonexistence of rational geodesic foliations in the torus, Mather sets and Gromov hyperbolic spaces

Rafael Ruggiero

Abstract. Given a rational homology class h in a two dimensional torus T^2, we show that the set of Riemannian metrics in T^2 with no geodesic foliations having rotation number h is C^k dense for every $k \in N$. We also show that, generically in the C^2 topology, there are no geodesic foliations with rational rotation number. We apply these results and Mather's theory to show the following: let (M, g) be a compact, differentiable Riemannian manifold with nonpositive curvature, if (M, g) satisfies the shadowing property, then (M, g) has no flat, totally geodesic, immersed tori. In particular, M has rank one and the Pesin set of the geodesic flow has positive Lebesgue measure. Moreover, if (M, g) is analytic, the universal covering of M is a Gromov hyperbolic space.

Keywords: Geodesic foliations, rotation number, Mather sets, shadowing property, Gromov hyperbolic.

Introduction

Let (M, g) be a C^∞, compact Riemannian manifold. We denote by $T_1 M$ the unit tangent bundle of M, $\tilde{M}$ denotes the universal covering of M, $\pi : \tilde{M} \longrightarrow M$ will be the covering map and $\pi_1(M)$ the fundamental group. Let $\phi_t : T_1 M \longrightarrow T_1 M$ be the corresponding geodesic flow. The notation $[a, b]$ represents a geodesic segment joining a and b. We say that (M, g) satisfies the shadowing property, if there exists a C^2 neighborhood V of (M, g) such that every metric $\bar{g} \in V$ has the following property: given a geodesic γ in $(\tilde{M}, \bar{g})$, there exist a geodesic β

Received 31 November 1999.
Partially supported by CNPq-GMD, FAPERJ, and the University of Freiburg.

in $(\tilde{M}, g)$ with finite Hausdorff distance from γ. The purpose of this article is to show the following result:

Theorem 1. *Let (M, g) be a compact Riemannian manifold with nonpositive curvature. If (M, g) satisfies the shadowing property then (M, g) contains no immersed, flat, totally geodesic tori.*

Combining Theorem 1 with some results of the theory of manifolds with nonpositive curvature, we obtain:

Theorem 2. *Let (M, g) be a compact manifold with nonpositive curvature. Assume that (M, g) has the shadowing property. Then*

1. *The rank of M is one, the Pesin set of the geodesic flow has positive Lebesgue measure, and in particular, the metric entropy of the geodesic flow with respect to the Lebesgue measure is positive.*

2. *If (M, g) is analytic, the universal covering $(\tilde{M}, g)$ endowed with the pullback of g is a Gromov hyperbolic space.*

Recall that $(\tilde{M}, g)$ is a Gromov hyperbolic space if every geodesic triangle is δ-thin for some $\delta > 0$, i.e., given x_0, x_1, x_2 in $\tilde{M}$, the distance from $p \in [x_i, x_{i+1}]$ to $[x_{i+1}, x_{i+2}] \cup [x_{i+2}, x_i]$ is bounded above by δ (here, the indices are taken mod. 3). The shadowing property is a dynamical counterpart of the stability of quasi-geodesics in Gromov hyperbolic spaces (see [9], [7]), an idea that goes back to Morse [13]. In some sense, the former is weaker than the latter because the shadowing property only requires the existence of shadows for quasi-geodesics which are geodesics of perturbations of the metric. In a previous paper [16], the conclusion of Theorem 1 is obtained for manifolds with nonpositive curvature satisfying the so called ϵ-C^k shadowing property, where $\epsilon = \frac{1}{5} inj.radius(M)$. It is not hard to check that the ϵ-C^k shadowing property implies the shadowing property, so Theorem 1 is a generalization of the results in [16]. The main idea of the proof is the following: if (M, g) contains a flat, immersed totally geodesic torus T^2, we perturb (M, g) so that T^2 remains totally geodesic and the new metric induced on T^2 is generic enough to allow us to apply Mather's work [12]. Then, Mather's results yield the existence of geodesics that cannot have shadows in the original metric, namely, geodesics "connecting", in the sense of Mather, many different asymptotic behaviors. Notice that perturbations preserving a flat, totally geodesic, immersed T^2 might not be generic when restricted to T^2, since they have to preserve the selfintersections of T^2 (which are also geodesics). I

am very grateful to Professor V. Bangert, who suggested me Theorem 1 while I was visiting Freiburg University, and encouraged me to write this paper.

1 Perturbations preserving totally geodesic submanifolds

Throughout the paper, all geodesics will be parametrized by arclength. The length of a compact, rectifiable curve C in (M, g) will be denoted by $l_g(C)$, the Hausdorff distance between two subsets A, B of a metric space (X, d) will be denoted by $d(A, B)$, and the distance in T_1M induced by (M, g) will be denoted by $d_{(T_1M,g)}$, or simply, d_{T_1M}. The main result of this section is a straightforward application of [16], Proposition 1.1.

Lemma 1.1. *Let (M, g) be a compact Riemannian manifold and let (T^2, g) be an immersed totally geodesic torus. Then, given an open, connected subset $B \subset T^2$ such that its closure $\bar{B}$ contains only simple points (i.e., $\bar{B}$ is disjoint from the set of points of selfintersection of T^2), an integer $k > 0$, a function $f : T^2 \longrightarrow R$ supported in an open subset of B such that $|f - 1|$ is C^k-close to 0, there exists a C^k conformal perturbation $(M, \bar{g})$ of (M, g), where $\bar{g}_p = \bar{f}(p)g_p$, such that*

1. *$(T^2, \bar{g})$ is totally geodesic.*
2. *The restriction of $\bar{f}$ to T^2 is the function f.*
3. *If the gradient ∇f is tangent to a g-geodesic segment γ contained in B, then γ is also a $\bar{g}$-geodesic segment.*

Proof. In fact, in [16], section 3, we find conditions for a function $\bar{f} : M \longrightarrow R$ to define such a conformal change $\bar{g}_p = \bar{f}(p)g_p$. We sketch the main ideas of the construction of the perturbations for the sake of completeness. In few words, every function $\bar{f} : M \longrightarrow R$ whose gradient is perpendicular to the tangent space of an embedded, totally geodesic submanifold preserves the totally geodesic character of the submanifold. This implies immediately item (3). To construct an extension of f satisfying items (1) and (2) we proceed as follows. For $q \in B$, let $N(q) \subset T_qM$ be the subspace of vectors which are orthogonal to T_qT^2. There exists $\delta > 0$ such that the open set

$$V = \cup_{q \in B}\{exp_q(v), \parallel v \parallel < \delta, v \in N(q)\}$$

meets T^2 precisely in B. Now, we can chose $\bar{f} : M \longrightarrow R$ such that

1. The support of $\bar{f}$ is V.

2. The restriction of $\bar{f}$ to B is the function f.

3. The restriction of $\bar{f}$ to $exp_q(N(q))$ is critical at $q \in B$ and has critical value $f(q)$.

The idea of the construction is very simple. Let $\Pi : V \longrightarrow B$ be the orthogonal projection along the fibers $exp_q(N(q))$. Consider a bump function $z : (-1, 1) \longrightarrow [0, 1]$ having a strict global maximum at $t = 0$, and define

$$\bar{f}(x) = f(\Pi(x))z(\frac{d_g^2(x, \Pi(x))}{\delta}).$$

It is easy to check that $\bar{f}$ is C^k small if f is C^k small. This shows items (1) and (2) in the lemma. □

2 Waists, homoclinic geodesics, and rational geodesic foliations

Recall that a geodesic γ in a Riemannian manifold (N, g) has conjugate points if there exists a pair of points $\gamma(t)$, $\gamma(s)$, where $t < s$, and a non trivial Jacobi field $J : [t, s] \longrightarrow TN$, with $J(t) = J(s) = 0$, $J(x) \neq 0$ for some $x \in [t, s]$. A geodesic has no conjugate points if and only if it minimizes the length of local variations joining any two points in it. We say that a geodesic γ in $(\tilde{M}, g)$ is globally minimizing (or a global minimizer) if it minimizes the distance between any two of its points. Morse in [13] called these geodesics class A geodesics, and it is clear that globally minimizing geodesics have no conjugate points. Globally minimizing geodesics in periodic metrics on R^2 have very special properties. For instance, such a geodesic projects onto a geodesic in the 2-torus with no self crossings, and posseses a rotation number, i.e., a well defined real homology class in $H_1(T^2, R)$. Throughout the paper, we shall identify rational homology classes with free homotopy classes in T^2. A closed geodesic α with nonzero free homotopy class will be called a *waist* if the length of any closed loop in its free homotopy class is strictly greater than the lenght of α. Waists in closed surfaces lift to global minimizers in the universal covering of the surface. Next, we list some basic properties of global minimizers in surfaces due essentially to Morse [13] and Hedlund [10]. Our basic reference for the subject is the work of Bangert [4], where we can find a complete, recent exposition of the theory with generalizations to the theory of monotone twist maps.

Lemma 2.1. *The set of vectors in T_1M which are tangent to globally minimizing geodesics of (M, g) is closed.*

Lemma 2.2. *Let (T^2, g) be a Riemannian structure on the torus T^2, and let α, β be two globally minimizing geodesics in (R^2, g). Suppose that $\alpha(0) = \beta(0) = p$. Then either $\alpha = \beta$, or $\alpha \cap \beta = p$ and* $\liminf_{t\to+\infty} d(\alpha(t), \beta) > 0$.

Lemma 2.3. *Let (T^2, g) be a Riemannian structure on T^2.*

1. *There exists a constant $D > 0$ such that every globally minimizing geodesic in (R^2, g) is contained in a tubular neighborhood of a straight line in the Euclidean plane R^2 of radius D.*
2. *Let γ be a closed geodesic which minimizes the length among closed loops in its free homotopy class. Let β be a geodesic in (T^2, g) whose lifts are global minimizers in (R^2, g) with rotation number ρ. Then the rotation number of γ is equal to ρ if and only if β does not cross γ.*

Given a closed geodesic $\alpha(t)$, $t \in [0, L]$ in (T^2, g), we say that a geodesic β is homoclinic to α if

$$\lim_{t\to+\infty} sup\{d(\alpha(t), \beta), d(\beta(t), \alpha)\} = 0,$$

and

$$\lim_{t\to-\infty} sup\{d(\alpha(t), \beta), d(\beta(t), \alpha)\} = 0.$$

The following result obtained by Morse [13] will be of fundamental importance for us.

Lemma 2.4. *Let α be a waist of (T^2, g), and let α_1, α_2 be two consecutive lifts of α in (R^2, g). Then there exist two globally minimizing geodesics β_1, β_2 in (R^2, g) such that*

$$\lim_{t\to+\infty} d(\alpha_1, \beta_1(t)) = 0, \quad \lim_{t\to-\infty} d(\alpha_2, \beta_1(t)) = 0$$

and

$$\lim_{t\to+\infty} d(\alpha_2, \beta_2(t)) = 0, \quad \lim_{t\to-\infty} d(\alpha_1, \beta_2(t)) = 0.$$

The previous Lemma has a sort of converse, that is straightforward from Morse's work.

Lemma 2.5. *Let α be a waist of (T^2, g). Then every globally minimizing geodesic of (R^2, g) contained in a strip bounded by two consecutive lifts of α behaves like the geodesics β_1 and β_2 in Lemma 2.4.*

Corollary 2.6. *Let α be a waist in (T^2, g), and assume that there exists a continuous foliation of T^2 by geodesics of (T^2, g) which have the same rotation number as α. Then every geodesic in this foliation is homoclinic to α. Moreover, any two geodesics in the foliation are asymptotic in the past and in the future (i.e, their distance goes to 0 as $t \to \pm\infty$).*

Proof. It is not hard to check that each geodesic in the foliation lifts to globally minimizing geodesics (see [17] for instance) in (R^2, g). Since the geodesics in the foliation have the same rotation number as α, by Corollary 2.4 these geodesics do not cross α. This implies that α must be a geodesic of the foliation. So Lemma 2.5 implies that each of these geodesics is homoclinic to α. Since there are essentially two possible asymptotic behaviours (described in Lemma 2.4) of the lifts of the geodesics in the foliation with respect to lifts of α; and two geodesics having different asymptotic behaviours always intersect, we easily deduce that the geodesics in the foliation have to be bi-asymptotic. □

3 The destruction of rational geodesic foliations by C^k-small bumps

Lemma 3.1. *Let α be a waist in (T^2, g), and let β be a geodesic that is homoclinic to α whose lifts are globally minimizing. Let V be any open subset such that the closure of V is disjoint from α and β. Then (T^2, g) can be approached in any C^k topology by a metric $(T^2, \bar{g})$ satisfying the following properties:*

1. *The geodesic α continues to be a waist of $(T^2, \bar{g})$ in its homotopy class, and β is still a $\bar{g}$-geodesic homoclinic to α whose lifts are $\bar{g}$-globally minimizing.*

2. *There is no $\bar{g}$-globally minimizing geodesic that is bi-asymptotic to some lift of β and meets $\pi^{-1}(int(V))$.*

Proof. Let $f : T^2 \longrightarrow R$ be a C^k perturbation of the function $h(p) = 1 \forall p \in T^2$ with support in V. Choose f such that $f(q) > 1$ for every $q \in int(V)$. Consider the metric $\bar{g}_p = f(p)g_p$ in T^2. The geodesic α continues to be a waist of $(T^2, \bar{g})$ in its homotopy class. Moreover, the curve β remains geodesic, homoclinic to α; and its lifts continue to be globally minimizing. This is simply because the conformal factor f increases arclength in V and the geodesics α, β do not cross V. Consider two consecutive lifts α_1, α_2 of α in $(R^2, \bar{g})$, that bound a strip F. Let us call also by $\beta \subset F$ a lift of the geodesic β in $(R^2, \bar{g})$. Suppose that there exists a $\bar{g}$-globally minimizing geodesic $\gamma \subset F$ that, at the same time, is bi-asymptotic to β and meets $\pi^{-1}(int(V))$. To simplify the notation, let us denote $\pi^{-1}(V)$ by V. Then there exists a continuous parametrization $s : R \longrightarrow R$ of γ such that

$$\lim_{t\to+\infty} d_{\bar{g}}(\beta(t), \gamma(s(t))) = 0, \quad \lim_{t\to-\infty} d_{\bar{g}}(\beta(t), \gamma(s(t))) = 0.$$

Given $\epsilon > 0$, let $t_\epsilon > 0$ be such that

$$sup_{h=g,\bar{g}} d_h(\beta(t), \gamma(s(t))) \leq \epsilon$$

for every $|t| \geq t_\epsilon$. Let us estimate the $\bar{g}$-length of $\gamma[s(-t), s(t)]$ in terms of its g-length. We have that

$$l_{\bar{g}}(\gamma[s(-t), s(t)]) = l_{\bar{g}}(\gamma[s(-t), s(t)] \cap V) + l_{\bar{g}}(\gamma[s(-t), s(t)] \cap V^c),$$

where V^c is the complement of V, which implies that

$$l_{\bar{g}}(\gamma[s(-t), s(t)]) = l_{\bar{g}}(\gamma[s(-t), s(t)] \cap V) + l_g(\gamma[s(-t), s(t)] \cap V^c).$$

Since f is strictly greater than 1 in V there exists $\rho > 0$ depending on γ such that $l_{\bar{g}}(\gamma[s(-t), s(t)] \cap V) = l_g(\gamma[s(-t), s(t)] \cap V) + \rho$. So we get

$$l_{\bar{g}}(\gamma[s(-t), s(t)]) = l_g(\gamma[s(-t), s(t)] \cap V) + \rho + l_g(\gamma[s(-t), s(t)] \cap V^c),$$

and hence

$$l_{\bar{g}}(\gamma[s(-t), s(t)]) = l_g(\gamma[s(-t), s(t)]) + \rho.$$

On the other hand, since β is g-globally minimizing, we have for $t \geq t_\epsilon$,

$$\begin{aligned} l_g(\beta[-t, t]) &\leq d_g(\beta(-t), \gamma(s(-t))) + d_g(\gamma(s(-t)), \gamma(s(t))) \\ &+ d_g(\gamma(s(t)), \beta(t)) \\ &\leq 2\epsilon + l_g(\gamma[s(-t)), s(t)]) \\ &\leq 2\epsilon + l_{\bar{g}}(\gamma[s(-t)), s(t)]) - \rho \end{aligned}$$

Therefore, since $l_g(\beta[-t, t])) = l_{\bar{g}}(\beta[-t, t])$ we conclude that

$$l_{\bar{g}}(\beta[-t, t]) - 2\epsilon + \rho \quad \leq \quad l_{\bar{g}}(\gamma[s(-t), s(t)]). \tag{1}$$

Take $\epsilon = \frac{\rho}{6}$. Let us use the notation $[p, q]$ to designate a minimizing geodesic segment joining p and q. Notice that the broken $\bar{g}$-geodesic η formed by

$$\eta = [\gamma(s(-t)), \beta(-t)] \cup [\beta(-t), \beta(t)] \cup [\beta(t), \gamma(s(t))],$$

joins the points $\gamma(s(-t))$ and $\gamma(s(t))$, and has $\bar{g}$-length

$$\begin{aligned} l_{\bar{g}}(\eta) &\leq l_{\bar{g}}(\beta[-t,t]) + 2\epsilon \\ &\leq l_{\bar{g}}(\beta[-t,t]) + \frac{\rho}{3} \end{aligned}$$

for every $t \geq t_\epsilon$. But inequality (1) implies that

$$l_{\bar{g}}(\beta[-t,t]) + \frac{2\rho}{3} \leq l_{\bar{g}}(\gamma[s(-t), s(t)])$$

for every $t \geq t_\epsilon$. Thus,

$$l_{\bar{g}}(\eta) \leq l_{\bar{g}}(\gamma[s(-t), s(t)]) - \frac{\rho}{3},$$

contradicting the $\bar{g}$-minimizing assumption on γ. This ends the proof of the lemma. □

Corollary 3.2. *Let $[\alpha]$ be a free homotopy class of (T^2, g) represented by a closed geodesic α whose length minimizes the length among closed loops in $[\alpha]$. Then (T^2, g) can be approached in any C^k topology by a metric $(T^2, \bar{g})$ such that*

1. *α is a waist in its homotopy class.*

2. *There is no foliation by geodesics with the same rotation number as α.*

Proof. Applying Lemma 1.1, we get C^k-arbitrarilly small conformal perturbations of (T^2, g) where the geodesic α is a waist in its homotopy class. Indeed, it suffices to take a conformal factor f supported in a small neighborhood B of a point in α that satisfies the following conditions:

1. The function f is strictly less than one in B,

2. The restriction of f to any geodesic segment in B that is normal to α reaches its minimum value at α.

In this way, the metric $\bar{g}_p = f(p)g_p$ satisfies the hypothesis of Lemma 1.1, (3) (the proof of this fact is made in detail in [16]). So, assume without loss of generality that α is a waist of (T^2, g). By the results of Section 2, a foliation of (T^2, g) by geodesics with the same rotation number of α is a foliation by geodesics which are homoclinic to α. Let η_1, η_2 be the projections in (T^2, g) of two g-globally minimizing geodesics β_1, β_2 which are bounded by two lifts of α, and that behave asymptotically like in Lemma 2.4. Given any open ball $W \in T^2$ whose closure is disjoint from α, the geodesics η_1, η_2 meet W only a finite number of times. So it is possible to find an open ball V whose closure is disjoint from α, η_1 and η_2. Then, Lemma 3.1 shows that we can approach (T^2, g) in any C^k topology by a metric $(T^2, \bar{g})$ without any foliation by geodesics homoclinic to α. □

Corollary 3.3. *The set of metrics in T^2 with no rational geodesic foliations is C^2 generic.*

Proof. Let $[\alpha]$ be a free homotopy class of (T^2, g) represented by a closed geodesic α whose length minimizes the length among closed loops in $[\alpha]$. We can perturb g conformally in the C^2 topology in order to transform α in a waist that is at the same time a closed, hyperbolic geodesic. This fact is straightforward from the results in [15] for instance, where the equation of the curvature of the perturbed metric is calculated explicitly. Denote this metric by $\bar{g}$. By Corollary 3.1, we can approach $(T^2, \bar{g})$ in any C^k topology by a metric with no foliations by geodesics with rotation number $[\alpha]$. Let M_0 be the set of metrics on T^2 with a hyperbolic waist in $[\alpha]$ and no geodesic foliations with rotation number $[\alpha]$. The point is that this set is open in the C^2 topology. In fact, let (T^2, g_0) be an element of M_0. There exists an open neighborhood V of g_0 in the C^2 topology where every metric in V has a hyperbolic waist in $[\alpha]$. We claim that there exists an open neighborhood $W \subset V$ of g_0 where no metric in W has foliations by geodesics with rotation number equal to the rotation number of α. For, suppose that there exists a sequence g_n of metrics in V approaching g_0 in the C^2 topology, where (T^2, g_n) has a foliation F_n with the above property. It is not hard to show that the sequence of foliations F_n converges to a foliation of (T^2, g_0) by globally minimizing geodesics with the same rotation number of α, because the set of foliations by globally minimizing geodesics is a co-compact subset of the collection of foliations of T^2. This clearly contradicts the assumption on g_0. We conclude that the set of metrics in T^2 with a waist on each homotopy class and no geodesic foliations with rational rotation number is the intersection of a countable collection of open, dense subsets of metrics. Thus, by Baire's Theorem, the above set is a nonempty subset of the first category in the set of metrics in T^2 endowed with the C^2 topology. □

Lemma 3.1 and Corollaries 3.1, 3.2, 3.3 hold for immersed, totally geodesic, two-dimensional tori in view of the results in Section 1. In fact, we can perform the conformal bumps required by the proof of Lemma 3.1 in an open subset of simple points in an immersed, totally geodesic two-dimensional torus, and then extend the perturbation as indicated in Lemma 1.1. So we have:

Corollary 3.4. *Let (M, g) be a complete Riemannian manifold with an immersed, totally geodesic two-dimensional torus (T^2, g). Let G_0 be the set of metrics in M for which the torus T^2 is totally geodesic. Then,*

1. *Given a nontrivial, free homotopy class $[\alpha]$ of T^2, the set of metrics in G_0 with a waist in $[\alpha]$ and no geodesic foliations with rational rotation number $[\alpha]$ is C^k-dense in G_0 for every $k \in N$.*

2. *The nonexistence of rational geodesic foliations in T^2 is C^2 generic in G_0.*

4 Birkhoff map and the existence of regions of instability

In this section, we denote by γ_θ, where $\theta = (p, v) \in T_1 T^2$, a geodesic with initial conditions $\gamma_\theta(0) = p$, $\gamma'_\theta(0) = v$. Given a Riemannian metric (T^2, h) in T^2, and a simple closed geodesic β, the Birkhoff map P_β associated to β is the first return map of geodesics intersecting β transversally. Namely, if γ is a geodesic in (T^2, h) with $\gamma(0) = \beta(s)$, and $\gamma'(0)$ is transversal to $\beta'(0)$, then $P_\beta(\gamma) = \gamma(r) = \beta(\bar{s})$ is the next intersection of γ with β. If we assume that such next intersection always exists, the Birkhoff map induces a map

$$T_\beta : S^1 \times (0, 2\pi) \longrightarrow S^1 \times (0, 2\pi),$$

$$T_\beta(s, t) = (\bar{s}, \bar{t}),$$

where $S^1 = R/Z$, (s, t) represents the geodesic $\gamma_\theta = \gamma_{\theta(s,t)}$ with initial conditions $\gamma_\theta(0) = \beta(sl_h(\beta))$, $t = cos^{-1} h(\gamma'_\theta(0), \beta'(sl_h(\beta)))$. So the coordinate t is a branch of the angle with respect to β. Analogously, $(\bar{s}, \bar{t})$ represents the first return of γ_θ to β, i.e., $P_\beta(\gamma_\theta) = \gamma_\theta(r) = \beta(\bar{s} l_h(\beta))$, and $\bar{t} = cos^{-1} h(\gamma'_\theta(r), \beta'(\bar{s} l_h(\beta)))$. This map might not be defined everywhere, and the purpose of this section is to show that perturbations of the Euclidean metric of the torus have well defined Birkhoff's maps satisfying some generic properties. Recall that a measure preserving map $F : S^1 \times R \longrightarrow S^1 \times R$, $F(t, s) = (\bar{t}, \bar{s})$, is called a twist map if $\frac{\partial \bar{t}}{\partial s} \neq 0$ for every point in the annulus $S^1 \times R$. There is a natural variational principle associated to twist maps whose minima are certain orbits of the dynamics satisfying very special properties, the so-called Aubry-Mather sets (for definitions and a clear, simple exposition of the theory we refer to [4]). Following Mather's notation, the collection of Aubry-Mather sets with rotation number ρ will be called $M_{F,\rho}$. One of the fundamental facts of the theory is that $M_{F,\rho}$ is nonempty for every ρ (see, for instance, [4]). When F is a Birkhoff map, homotopically nontrivial invariant curves in $M_{F,\rho}$ correspond to invariant tori in T_1T^2 foliated by globally minimizing geodesics. A Birkhoff region of instability B of a twist map F of the annulus is an invariant closed annulus bounded by two invariant, noncontractible curves, such that there is no other noncontractible invariant curve in its interior. To simplify our notation, we shall often identify geodesics with their corresponding orbits in a Birkhoff map. The main result of the section is the following:

Lemma 4.1. *Let (T^2, g) be a flat, immersed, totally geodesic torus in (M, g), let $p \in T^2$ be a simple point, and γ_θ, $\theta = (p, v)$, be a closed geodesic that minimizes the length among closed loops in its free homotopy class. Given $\delta > 0$, there exits a δ-C^k perturbation $(M, \bar{g})$ of (M, g) with the following properties:*

1. *$(T^2, \bar{g})$ is totally geodesic.*
2. *There exist a closed, $\bar{g}$-geodesic β in $(T^2, \bar{g})$, and a locally isometric covering $\Pi : (\bar{T}^2, \bar{g}) \longrightarrow (T^2, \bar{g})$, with the following properties:*
   - *The set $\bar{T}^2$ is a 2-dimensional manifold, and the (unique) lift $\bar{\beta}$ of β in $\bar{T}^2$ is a simple closed geodesic.*
   - *The Birkhoff map $T_{\bar{\beta}}$ associated to $\bar{\beta}$ is a well defined twist map of an annulus C.*
   - *The annulus C posseses a Birkhoff region of instability B which contains the lift of γ_θ in $\bar{T}^2$, and every Mather set $M_{T_{\bar{\beta}},\rho}$, where ρ is the rotation number of γ_θ with respect to $T_{\bar{\beta}}$.*

Notice that the annulus C is not in general in the standard form $S^1 \times [a, b]$. We shall usually identify an orbit of the Birkhoff map with the underlying geodesic. We show first some elementary lemmas about conjugate points and twisting of the Birkhoff map.

Lemma 4.2. *Let (T^2, h) be a Riemannian structure on T^2. Suppose that there exists a simple closed geodesic β of (T^2, h) with the following properties:*

1. *The associated Birkhoff map induces a well-defined map on an annulus $T_\beta : C \longrightarrow C$.*
2. *There exists $\lambda > 0$ with the following property:*

   Given a geodesic γ_θ represented by the Birkhoff map, with $\gamma_\theta(0) \in \beta$, let r_θ be the first return time of γ_θ to β (i.e., $\gamma_\theta(r_\theta) = P_\beta(\gamma_\theta)$). Then the geodesic segment $\gamma_\theta[-\lambda, r_\theta + \lambda]$ has no conjugate points.

Then the Birkhoff map of β is a twist map.

Proof. Since this lemma is well-known we just sketch a proof for the sake of completeness. On the one hand, the Birkhoff map preserves the measure $sin(t)dsdt$ of the annulus [6], where s, t are the coordinates defined in the beginning of the section. On the other hand, the absence of conjugate points in the geodesic segment $\gamma_\theta[-\lambda, r_\theta + \lambda]$ implies the convexity of the generating function H of the Birkhoff map at the pair $(s(\gamma_\theta(0)), s(\gamma_\theta(r_\theta)))$ (see [4]). This is equivalent to the twist property of the map. □

Lemma 4.3. *Let (T^2, g) be a flat metric on T^2. Given $k \in N$, there exists $\delta > 0$, $\delta' > 0$, such that every δ-C^k perturbation $(T^2, \bar{g})$ of (T^2, g) satisfies the following property:*

Let $S \subset T_1T^2$ be a region invariant by the geodesic flow of $(T^2, \bar{g})$, whose boundary consists on two invariant tori foliated by globally minimizing geodesics which are within a (Hausdorff) distance δ'. Then there exists a closed $\bar{g}$-geodesic β whose associated Birkhoff map restricted to a certain annulus C represents the geodesic flow restricted to S. Moreover, the Birkhoff map is a twist map.

Proof. Recall that every flat torus has a pair of generators of the fundamental group v_1, v_2 represented by simple closed geodesics which meet making an angle of at least $\frac{\pi}{3}$ (see [16] for instance). Let γ_1, γ_2 be two straight lines in the plane tangent to v_1, v_2 respectively, with $\gamma_1(0) = \gamma_2(0)$. Then it is clear that every straight line intersects either γ_1 or γ_2 forming an angle of at least $\frac{\pi}{6}$. These geometric features of fundamental domains of an Euclidean torus persist in some sense under small perturbations of the metric. In fact, let now γ_1, γ_2 be two closed $\bar{g}$-geodesics in T^2 with free homotopy classes $[\gamma_i] = v_i$. Suppose that $\gamma_1(0) = \gamma_2(0)$. Given $k \in N$, $\epsilon > 0$, there exists $\delta > 0$ such that for every δ-C^k perturbation $(T^2, \bar{g})$ of (T^2, g), we have that the angle of intersection between γ_1 and γ_2 is at least $\frac{\pi}{3} - \epsilon$. Hedlund observed in [10] that any globally minimizing geodesic in (R^2, g) whose rotation number differs from v_i must cross all the lifts of γ_i.

Claim: There exists $A > 0$, $A = A(\delta)$, such that every metric $(T^2, \bar{g})$ that is δ-C^k close to an Euclidean metric, where $k \geq 1$, has the following property:

Let β be a geodesic of $(T^2, \bar{g})$ that lifts to a globally minimizing geodesic $\bar{\beta}$ in $(R^2, \bar{g})$ with rotation number $[\beta]$. Then, either the time spent by $\bar{\beta}$ between any two consecutive lifts of γ_1 is less than A, or the time spent by $\bar{\beta}$ between any two consecutive lifts of γ_2 is less than A.

Otherwise, there would exist a sequence of metrics g_n, $n \in N$, which are C^k close to g, geodesics β_n of (T^2, g_n) that lift to globally minimizing g_n-geodesics $\bar{\beta}_n$ in R^2 having rotation numbers $[\beta_n]$, a sequence of intervals $[t_{n,i}, s_{n,i}]$, $i = 1, 2$ with $|t_{n,i} - s_{n,i}| \to +\infty$, such that the geodesic segment $\bar{\beta}_n[t_{n,i}, s_{n,i}]$ is contained in a strip bounded by two consecutive lifts of $\gamma_{n,i}$. Here, $\gamma_{n,i}$ is a closed minimizing g_n-geodesic with homotopy class v_i. Now, since the periods of the geodesics $\gamma_{n,i}$ are uniformly bounded by some constant P, and the angles between $\gamma_{n,1}, \gamma_{n,2}$ are close to $\frac{\pi}{3}$; we can easily show that there exists n_0 such that the geodesic β_n has self-crossings for every $n \geq n_0$. Since globally minimizing geodesics cannot have self-crossings, this yields a contradiction.

The Claim implies that there exists $a = a(A) > 0$ such that for any $\delta - C^k$ small perturbation $(T^2, \bar{g})$ of (T^2, g), and any $\bar{g}$-geodesic β whose lifts are globally minimizing, then either each time that β and γ_1 cross they form an angle greater than a, or each time β and γ_2 cross they form an angle greater than a.

Now, given $\epsilon > 0$, let $\delta' = \delta'(\epsilon, A) > 0$ be such that for every C^k-δ small perturbation $(T^2, \bar{g})$ of the Euclidean (T^2, g) we have

1. Given two $\bar{g}$-geodesics γ_θ, γ_σ, where $\theta, \sigma \in (T_1T^2, \bar{g})$, satisfying $d_{T_1T^2}(\theta, \sigma) < \epsilon$, then $d_{\bar{g}}(\gamma_\theta(t), \gamma_\sigma(t)) \leq \delta'$ for every $|t| \leq 4A$;

2. Given a $\bar{g}$-geodesic γ_θ which has no conjugate points in the interval $(-1, 4A)$, then every $\bar{g}$-geodesic γ_σ with $d_{\bar{g}}(\gamma_\theta(t), \gamma_\sigma(t)) \leq \delta'$ for every $-1 \leq t \leq 4A$ has no conjugate points in the interval $(-\frac{1}{2}, 3A)$;

3. If a $\bar{g}$-geodesic segment $\gamma_\theta[0, T]$, where $|T| \leq A$, crosses γ_i at the points $\gamma_\theta(0)$, $\gamma_\theta(T)$ with angles greater than a, then any $\bar{g}$-geodesic $\gamma_\sigma[-\frac{A}{2}, 2A]$ satisfying $d_{\bar{g}}(\gamma_\theta(t), \gamma_\sigma(t)) \leq \delta'$ for every $-\frac{A}{2} \leq t \leq 2A$ also crosses γ_i at (at least) two points, and the angles formed by $\gamma_\theta[0, T]$ and γ_i at their crossings is greater than $\frac{a}{2}$.

Lemma 4.3 is straightforward from the above Claim. Indeed, let δ, δ' be the numbers previously defined, let S be a region in $(T_1T^2, \bar{g})$ bounded by two invariant tori foliated by globally minimizing geodesics with rotation numbers r_1, r_2. Let F_1, F_2 be the projections of these foliations in T^2. Assume that the boundary tori of S are within a Hausdorff distance δ'. By the Claim, we can assume that F_1 crosses γ_1 forming angles of at least a, and then the first return map P of F_1 to the cross section γ_1 is well defined, as well as all of its iterates. By the choice of δ', the same happens to the projections in T^2 of every orbit of the $\bar{g}$-geodesic flow contained in S. Moreover, the return times of the geodesics in S is bounded above by $3A$, and they meet γ_1 with angles greater than $\frac{a}{2}$. Hence, the map P_{γ_1} is well defined for every projection of an orbit in S. By item 2 above and the proof of Lemma 4.2, (2), the map P_{γ_1} restricted to the projections of the orbits in S gives rise to a twist map T_{γ_1}. We have to check that T_{γ_1} is defined in an annulus. Observe that there is a bijective correspondence between the above geodesics and an annulus. Indeed, let $t(\eta(0), \eta'(0))$ be the angle coordinate defined in the beginning of the section, where η is the projection of some orbit in S satisfying $\eta(0) \in \gamma_1$. Let $F_i(p)$, $i = 1, 2$, be the geodesic of F_i passing through p, and denote by $t_i(s)$, $s \in R/Z$, the angle coordinate of the geodesic $F_i(p)$ at the point $p = \gamma_1(sl(\gamma_1))$. Assume that $t_1(s) < t_2(s)$ for every $s \in R/Z$. It is not hard to see that the twist property of the map T_{γ_1} implies that $t(\eta(0), \eta'(0)) \in [t_1(s(\eta(0))), t_2(s(\eta(0)))]$ for every projection η of an orbit in S; and hence, the set of coordinates (s, t) of the projections of the orbits in S is the set

$$C = \bigcup_{s \in R/Z} [t_1(s), t_2(s)]$$

This set is homeomorphic to an annulus, its boundary consists on the two closed curves $C_1 = \{(s, t_1(s)), s \in R/Z\}$, and $C_2 = \{(s, t_2(s)), s \in R/Z\}$. This finishes the proof of Lemma 4.3. □

Proof of Lemma 4.1: Given $k \geq 5$, let $\delta > 0$, δ' be the numbers defined in Lemma 4.3. Let $(T^2, \bar{g})$ be δ-C^k close to an Euclidean torus (T^2, g). By KAM theory, there exists $\delta_0 > 0$ such that for every $\delta \leq \delta_0$ there exists a lamination $F_{\bar{g}} \subset T_1 T^2$ whose leaves are invariant tori. In each invariant torus the dynamics of the geodesic flow is equivalent to an irrational flow of the torus, and the Lebesgue measure of $F_{\bar{g}}$ tends to be full as δ goes to 0. Any such torus gives rise to a foliation by $\bar{g}$-geodesics of T^2 with irrational rotation number. Moreover, according to [4], any geodesic foliation with irrational rotation number is unique with this rotation number. Given $\delta'' > 0$, we can choose δ very small so that the boundary tori of every connected component of the complement of $F_{\bar{g}}$ are within a Hausdorff distance less than δ''. Then, by Lemma 4.3, letting $\delta'' \leq \delta'$, every invariant region in $(T_1 T^2, \bar{g})$ whose interior has no invariant tori determines a Birkhoff map in an annulus that is a twist map.

Now, let us suppose that (T^2, g) is a flat, totally geodesic torus immersed in (M, g). Let $p \in T^2$ be a simple point, let γ_θ be a closed geodesic through p, and let $(T^2, \bar{g})$ be a δ'-C^5 perturbation of (T^2, g) satisying the hypothesis of Corollary 3.4, (1), with $\alpha = \gamma_\theta$. Since (T^2, g) is flat, it is covered by flat planes in $(\tilde{M}, g)$. Let (P, g) be one of these planes. Let us remind that $(T^2, \bar{g})$ and (T^2, g)

correspond to the same torus immersion (by the properties of the perturbation $\bar{g}$), so $(P, \bar{g})$ is a totally geodesic plane in $(\tilde{M}, \bar{g})$ that covers $(T^2, \bar{g})$. Let ω, η be two closed $\bar{g}$-geodesics whose homotopy classes generate $\pi_1(T^2)$. There exists a pair of covering transformations D_ω, D_η, acting on $(P, \bar{g})$ by isometries, whose axes are the lifts of ω and η respectively in $(P, \bar{g})$. Now, the quotient $P/ < D_\omega, D_\eta >$ of P by the action of D_ω, D_η is a Riemannian torus $\bar{T}^2$ which is locally isometric to $(T^2, \bar{g})$. Moreover, Lemmas 4.2 and 4.3 apply to $\bar{T}^2$, since $\bar{T}^2$ is close to a flat torus. Hence, we can assume without loss of generality that $(T^2, \bar{g})$ is embedded in $(M, \bar{g})$.

We claim that γ_θ is contained in a Birkhoff region of instability. Indeed, let ρ be the rotation number of γ_θ in $H_1(T^2, R)$. By Corollary 3.4, there is no geodesic foliation in $(T^2, \bar{g})$ with rotation number ρ. This implies that θ belongs to the complement of the closure of invariant tori in $T_1(T^2, \bar{g})$, because the collection of invariant tori is closed in $T_1(T^2, \bar{g})$. In particular, θ belongs to the complement of $F_{\bar{g}}$, and there exists an open neighborhood V of θ in this complement. Hence, the connected component Γ of the complement of the set of invariant tori that contains θ is bounded by two invariant tori which are within a Hausdorff distance less than δ'. Therefore, by Lemmas 4.2, 4.3, there exists a closed $\bar{g}$-geodesic β such that the geodesic flow restricted to Γ is represented by the Birkhoff map $T_\beta : C \longrightarrow C$ associated to β, where C is some annulus. Moreover, the boundary of C consists on two invariant curves representing the invariant tori in the boundary of Γ, and there are no other invariant curves in the interior of Γ. This finishes the proof of Lemma 4.1.

5 The proof of the main Theorem

We first recall the following property of twist maps of the annulus whose proof is due to J. Mather ([12], Theorem 4.2).

Theorem 5.1. *Let $F : S^1 \times R \longrightarrow S^1 \times R$ be a monotone twist map, and let B be a Birkhoff region of instability. Let Γ_-, Γ_+ be the invariant curves in the boundary of B, with rotation numbers $\rho_- < \rho_+$. Consider for each $i \in Z$ a real number $\omega_i \in [\rho_-, \rho_+]$ and a positive number ϵ_i. Then there exists an orbit $\{F^k(x)\}$ in B and an increasing bi-infinite sequence $\{j_i\}$ of integers such that $d(F^{j_i}(x), M_{F,\omega_i}) < \epsilon_i$.*

Combining Lemma 4.1 and Theorem 5.1 we get

Corollary 5.2. *Let $k \geq 5$, let $\delta > 0$ and $(T^2, \bar{g})$ be the δ-C^k perturbations of an Euclidean metric (T^2, g) defined in Lemma 4.1. Then there exist two different* *rotation numbers v_1, $v_2 \in H_1(T^2, R)$, geodesics η, $\gamma_{1,n}$, $\gamma_{2,n}$ in $(R^2, \bar{g})$, where $n \in N$, satisfying:*

1. *The geodesics $\gamma_{i,n}$ are globally minimizing with rotation numbers $[\gamma_{i,n}] = v_i$, for $i = 1, 2$.*
2. *There exists a sequence t_k, $k \in Z$, with $t_k \to +\infty$, such that*

$$d_{T_1 T^2}(\eta(t_{2n}), \gamma_{1,n}) < |\frac{1}{2n}|,$$

and

$$d_{T_1 T^2}(\eta(t_{2n+1}), \gamma_{2,n}) < |\frac{1}{2n+1}|.$$

Proof. Let F be the Birkhoff map associated to $(T^2, \bar{g})$ obtained in Lemma 4.1, and let B be its region of instability. Let Γ_-, Γ_+ be the invariant curves in the boundary of B whose rotation numbers are $\rho_- < \rho_+$, and let γ_θ be the closed geodesic defined in Lemma 4.1. Let α be the rotation number of the orbit of F associated to the geodesic γ_θ. Since there is no invariant curve in B whose rotation number is α, we know that ρ_- and ρ_+ must be different from α. We can apply Theorem 5.1 for instance to the sequences $\omega_{2i} = \rho_-$, $\omega_{2i+1} = \alpha$, $\epsilon_k = \frac{1}{k}$. By the construction of the Birkhoff map, this yields the existence of geodesics $\gamma_{1,n}$, $\gamma_{2,n}$, η in the universal covering of $(T^2, \bar{g})$, where $\gamma_{1,n}$, $\gamma_{2,n}$ have rotation numbers $[\gamma_{1,n}] = v_1$, $[\gamma_{2,n}] = [\gamma_\theta]$; and $\eta(t)$ approaches to within $|\frac{1}{2n}|$ or $|\frac{1}{2n+1}|$ to, alternatively, $\gamma_{1,n}$ and γ_{2n}. This finishes the proof of the Corollary. □

Proof of Theorem 1: The proof of Theorem 1 is by contradiction. Let (M, g) be a compact manifold with nonpositive curvature satisfying the shadowing property. Assume that (M, g) contains a flat, totally geodesic, immersed torus T^2. Let $(M, \bar{g})$ be a small C^k perturbation of (M, g) satisfying Lemma 4.1, i.e., the torus $(T^2, \bar{g})$ is totally geodesic, and the restriction of the geodesic flow of $(M, \bar{g})$ to $(T^2, \bar{g})$ posseses an invariant region whose associated Birkhoff map has a region of instability. Notice that there exist infinitely many flat, totally geodesic planes in $(\tilde{M}, g)$, that cover (T^2, g), which are totally geodesic in $(\tilde{M}, \bar{g})$ and cover $(T^2, \bar{g})$. So let us choose one of these flats P, and let $\gamma_{1,n}$, $\gamma_{2,n}$, for $n \in N$, and η, be the geodesics in P given by Corollary 5.1. By Hedlund's work [10], there exists a constant $D > 0$ such that each geodesic $\gamma_{i,n}$ is contained in a tubular neighborhood of radius D of a straight line $L_{i,n}$ in P. The rotation number of

$L_{i,n}$ is clearly v_i, the rotation number of $\gamma_{i,n}$, for every $n \in N$. So the lines $\{L_{1,n}, n \in N\}$ are parallel to each other, as well as the lines $\{L_{2,n}, n \in N\}$. Let us call by η_0 the corresponding shadow in $(\tilde{M}, g)$ of the geodesic η. By the definition of the shadowing property, there exists $A > 0$ such that η is contained in a g-tubular neighborhood of radius A of η_0. Since η is a subset of the flat P, which is totally geodesic in $(\tilde{M}, g)$, the geodesic η_0 is within a distance A from P. By nonpositive curvature geometry, there exists a flat strip F, whose width is less than A, bounded by η_0 and a g-geodesic $\hat{\eta}_0$ in P. The strip F meets P perpendicularly precisely at the points of $\hat{\eta}_0$. But P is flat, therefore the geodesic $\hat{\eta}_0$ is a straight line. Hence, the $\bar{g}$-geodesic η is within a distance $2A$ from a straight line $L = \hat{\eta}_0$ in P. By Corollary 5.1, given $\epsilon > 0$, there are sequences $t_{i,n} < s_{i,n}$, for $i = 1, 2$, $n \in N$, where $\lim_{n\to+\infty} s_{i,n} - t_{i,n} = +\infty$, such that

$$d(\eta[t_{i,n}, s_{i,n}], \gamma_{i,n}) < \epsilon,$$

for every $n \in N$. This implies that

$$d(\eta[t_{i,n}, s_{i,n}], L_{i,n}) < \epsilon + D$$

for every $n \in N$. Hence, there exist sequences of points $a_{i,n}$, $b_{i,n}$ in L, where $\lim_{n\to+\infty} d(a_{i,n}, b_{i,n}) = +\infty$, such that

$$d([a_{i,n}, b_{i,n}], L_{i,n}) \leq \epsilon + D + 2A.$$

Here, $[a, b]$ means the subsegment of L joining two points $a, b \in L$. This clearly leads to a contradiction, because given two line fields v_1, v_2 in P, the straight line L makes a constant angle α_1 with the field v_1, and a constant angle α_2 with the field v_2. Since the contradiction arose from the assumption of the existence of a flat, totally geodesic, immersed torus in (M, g), we get Theorem 1.

Theorem 2 follows from Theorem 1 and two classical results of the theory of manifolds of nonpositive curvature. By [2], [3], manifolds of rank ≥ 2 are foliated by flats and have a dense subset of flat tori. So Theorem 1 implies that the geodesic flow of a rank m manifold, $m \geq 2$, cannot satisfy the shadowing property. Standard arguments in nonpositive curvature geometry imply that the Pesin set of the geodesic flow has positive Lebesgue measure and thus, the metric entropy with respect to the Lebesgue measure is positive. On the other hand, if (M, g) is analytic, by [5], the existence of a flat, totally geodesic plane in $\tilde{M}$ implies the existence of a flat immersed torus in M. Finally, Eberlein's characterization of visibility manifolds of nonpositive curvature through the non-existence of flat planes in $\tilde{M}$ [8], tells us that the shadowing property implies the visibility of $(\tilde{M}, g)$. Now, it is easy to verify that $(\tilde{M}, g)$ is a visibility manifold if and only if it is a Gromov hyperbolic space.

References

1. Arnold, V. I.: Mathematical Methods of Classical Mechanics. Second Edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, Berlin, Heidelberg.
2. Ballmann, W., Brin, M., Eberlein, P.: Structure of manifolds with nonpositive curvature I. Annals of Math. 122 (1985) 171-203.
3. Ballmann, W., Brin, M., Spatzier, R.: Structure of manifolds with nonpositive curvature II. Annals of Math. 122 (1985) 205-235.
4. Bangert, V.: Mather sets for twist maps and geodesics on tori. Dynamics reported, Vol. 1. U. Kirchgraber, H. O. Walter Editors, 1988 John Wiley and Sons and B. G. Teubner.
5. Bangert, V., Schröeder, V.: Existence of flat tori in analytic manifolds of nonpositive curvature. Ann. Sci. Ec. Norm. Sup. 24 (1991) 605-634.
6. Birkhoff, G. D.: Surface transformations and their dynamical applications. Acta Math., 43 (1922), 1-119.
7. Bonk, M.: Quasi-geodesic segments and Gromov hyperbolic spaces. Geom. Dedicata 62 (1996) 281-298.
8. Eberlein, P.: Geodesic flows in certain manifolds without conjugate points. Trans. Amer. Math. Soc. 167 (1972) 151-170.
9. Gromov, M.: Hyperbolic groups. Essays in group theory 75-263. Gersten Ed. Springer-Verlag, New York.
10. Hedlund, G.: Geodesics on a two dimensional Riemannian manifold with periodic coefficients. Annals of Math. 33 (1932) 719-739.
11. Mather, J.: Destruction of invariant circles. Ergod. Th. and Dynam. Sys. (1988), 8, 199-214. Charles Conley Memorial Volume.
12. Mather, J.: Variational construction of orbits of twist diffeomorphisms. J. Amer. Math. Soc. 4, 2 (1991) 207-263.
13. Morse, M.: A fundamental class of geodesics on any closed surface of genus greather than one. Trans. Amer. Math. Soc. 26 (1924) 25-60.
14. Moser, J. K., Siegel, C. L.: Lectures on Celestial Mechanics. Springer-Verlag, New York, Heidelberg, Berlin, 1971.

15. Ruggiero, R.: On the creation of conjugate points. Math. Zeitschrift (1991), 208, 41-55.
16. Ruggiero, R.: Topological stability and Gromov hyperbolicity. Erg. Th. Dyn. Sys. (1999), 19, 143-154.
17. Ruggiero, R.: On manifolds admitting continuous foliations by geodesics. Geometriae Dedicata 78: 161-170, 1999.

Rafael Ruggiero
PUC-Rio
Departamento de Matemática
Rua Marques de São Vicente 225
Gávea, Rio de Janeiro, Brasil
E-mail: rorr@mat.puc-rio.br